# The Use of Change Identifiers to Update Footprints of Dot Patterns in Real Time

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**Abstract.** Commonly, in the field of spatial knowledge representation, there is a need to assign to a group of individual entities, considered as an aggregate, a spatial location known as its 'footprint'. The problem of finding an appropriate footprint for an aggregate in a static context has been fairly thoroughly researched, but little thought has been given to possible changes of the footprint over time resulting from the movement of individuals into, out of, or within the aggregate. For many practical applications, it is required to track the footprint of a 'live' dynamic system such as a crowd or flock. This paper looks at the problems involved in maintaining footprints over non-static dot patterns and how to negotiate the trade-offs between efficiency of computation and accuracy of result. The key notion is to use 'change identifiers' to determine when and how often the footprint of a moving aggregate should be updated. Preliminary results from an implemented system are presented.

## 1 Introduction

In spatial information theory one often encounters the problem of representing groups or aggregates, which at a fine level of granularity appear as pluralities with a scattered distribution but at a coarser granularity may be treated as single coherent individuals with their own behaviour and properties. Familiar examples from the everyday world include forests (i.e., aggregates of trees), flocks and crowds (aggregates of animals or people), and conurbations (aggregates of buildings).

In recent research, attention has been paid to the problem of assigning a spatial location to an aggregate considered as a unit, given as inputs the spatial locations of the individual components [5, 9]. In abstract form, the two-dimensional problem is, given a set of *dots* (i.e., objects sufficiently compact to be idealised as points) in the plane, to determine a *footprint* representative of the spatial distribution of the collection of dots taken as a whole. The footprint will be a two-dimensional region, which, depending on one's purposes, may be required to satisfy various contraints such as polygonality, connectedness, topological regularity, convexity, etc [6]. The problem generalises to three dimensions in the obvious way, but for simplicity the discussion in this paper will be restricted to the two-dimensional version.

It cannot be too strongly emphasised that there does not exist a uniquely "best" or "correct" footprint for a given dot pattern. In [8] it was shown experimentally that footprints selected as "good" by human subjects represent optimal trade-offs between the conflicting goals of minimising the area and minimising the perimeter, but this certainly does not tell the whole story. For human purposes, an important feature of a good footprint is that it "looks" right, that is, it represents a shape that we "see" in the dot pattern itself. But for some purposes, one might prefer to use a footprint that is very easily computed (e.g., the minimal axis-aligned bounding rectangle) or which has well-known mathematical properties (e.g., the convex hull), even though in many cases these do not provide a close visual match to the dot pattern.

Many different algorithms for generating footprints from dot sets have been proposed, in contexts such as geographical information theory [1, 9], pattern recognition [13, 5], computer vision [11], and computational geometry [7], to cite only a few representative examples. In all these cases, however, the assumption is that the dot patterns are *static*. In reality, many examples of collectives or aggregates are dynamic, with either the location or the membership, or both, varying over time [16]. Of our examples above, flocks and crowds vary in both these respects over a short time scale; forests and conurbations also vary, but the time-scale is typically several orders of magnitude greater. There has been some work into tracking aggregates but the majority of this has been centered around object tracking within video (where the pixels are the dots) such as in [4] and [2], however there is a difference between tracking a fixed shape amongst background noise and maintaining a footprint of a possibly changing shape. There is also the work performed by [12] concerning tracking herds, this is less interested in the footprint of the aggregate and focuses on the herd as an abstract looking at on four major possible evolvements: *expand, join, shrink and leave*.

The problem we address in this paper is how to track the footprint of dynamically changing aggregates of dots. In the case of fast-moving aggregates, an added constraint is that the tracking should take place in real time. Footprint algorithms typically run in time $O(n \log n)$ or worse (sometimes much worse), where $n$ is the number of dots. Hence recomputing the footprint *ab initio* every time there is a change in the dot pattern will be computationally costly, making real-time recomputation infeasible in many cases. Importantly we have limited the data we expect to simply the coordinate positions of the dots, this is to keep the system as general as possible, although information such as identity may allow for extensions to the change identifiers.

One possible approach would be to look for a way to *update* the footprint incrementally rather than recompute it entirely. In an ideal world, one could do this in such a way that the footprint assigned to the dots at any time is always identical to the footprint that would be obtained if it were recomputed. In general, for most types of footprint it is unlikely that such exact tracking can be accomplished with significantly less cost than recomputing the footprint every time.

Instead, we propose a method by which the position of the dots

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in relation to the most-recently computed footprint is continuously monitored, and the footprint is only recomputed when the mismatch between the dot positions and the current footprint exceeds some preassigned threshold of accuracy. Clearly there will be a trade-off between the level at which the accuracy threshold is set and the resultant frequency of recomputation, and we investigate the nature of this trade-off with a view to optimising it.

Our approach is to use a suite of easily computable *change identifiers*, each with its own threshold. Recomputation of the footprint is triggered when some aggregate value computed from the values returned by the change identifiers exceeds a given threshold. In the simplest form this aggregate value could be a count of how many of the change identifiers individually exceed their thresholds, amounting in effect to a vote amongst the change identifiers. Alternatively, the change identifiers could be ranked in order of importance and a weighted combination of their values compared with some threshold. We investigate the effect of using different sets of change identifiers, and different ways of combining the results returned by them.

The plan for the remainder of this paper is as follows. In §2 we fix some terminology and formalise the approach described above in the form of an algorithm presented in pseudocode. In §3 we discuss a range of possible change identifiers, evaluating them in terms of their ease of computation and informativeness in relation to the task in hand. In §4 we consider combinations of change identifiers, and discuss the computation of aggregate values and thresholds. In §5, we provide a theoretical analysis of the kinds of results to be expected from running the algorithm, and in §6 we describe the current state of our implementation and some preliminary results. Finally, in §7 we summarise the results obtained so far and outline our plans for future work.

## 2 Process

The basic process we implement is shown as Algorithm 1, which works as follows. The incoming data consists of a sequence of dot patterns (which might come from, e.g., observations relayed by sensor arrays). The dot pattern associated with time step $i$ is denoted $DP_i$, and is referred to as the **current dot pattern** when $i$ is the current time. An algorithm for generating footprints from dot patterns is assumed given (we shall refer to this as the **footprint algorithm**), and at the beginning of the sequence a footprint $f(DP_0)$ is generated for dot pattern $DP_0$ and saved as the **stored footprint** $SFP_0$. The dot pattern $DP_0$ from which it is generated is stored as the **stored dot pattern** ($SDP_0$). At subsequent time steps, the change identifiers are used to determine whether a new footprint should be computed; this is done by evaluating the extent to which the current dot pattern $DP_i$ differs from the previously stored dot pattern $SDP_{i-1}$. If this value, $eval(DP_i, SDP_{i-1}, SFP_{i-1})$, exceeds some pre-set threshold, then a new footprint $f(DP_i)$ is generated as the new stored footprint $SFP_i$, and the current dot pattern is used as the new stored dot pattern $DP_i$. Otherwise, the stored dot pattern and footprint are retained from the previous time step. For any dot pattern $DP_i$, the footprint $f(DP_i)$ that would be computed from it (whether or not this computation actually takes place) will be referred to (admittedly somewhat tendentiously, bearing in mind the non-uniqueness of the footprint) as the **true footprint** for that dot pattern.

## 3 Change Identifiers

Each change identifier returns a value representing some measure of change. To produce this value it has access to the stored dot pattern, the current dot pattern and the stored footprint. Most of the identifiers listed below do not use the stored footprint; this enables them to be used in conjunction with a wide range of footprint algorithms, since they make no assumptions concerning the nature of the footprint (e.g., whether it must be polygonal, can have holes or multiple components, etc.). To assess whether the value it returns should force a footprint update, a threshold is associated with each change identifier; and if change identifiers are to be combined, a method to normalise their values is required. These ideas are discussed in the section on change identifier sets (§4). We describe the case where a change identifier exceeds its threshold as a 'failure' since in this case the stored footprint is deemed to have failed to represent the current dot pattern accurately.

**Algorithm 1** Main Process

```
1:  i = 0
2:  Input first dot pattern DP_0
3:  SFP_0 = f(DP_0)
4:  SDP_0 = DP_0
5:  repeat
6:     i = i + 1
7:     Input DP_i
8:     if eval(DP_i, SDP_{i-1}, SFP_{i-1}) > threshold then
9:        SDP_i = DP_i
10:       SFP_i = f(DP_i)
11:    else
12:       SDP_i = SDP_{i-1}
13:       SFP_i = SFP_{i-1}
14:    end if
15: until No more input available
```

The identifiers listed below are not exhaustive and we are not presenting them as a definitive final set, the do however cover a range of possible transformation types the dot pattern could undergo, e.g., changes in position, changes in distribution, and changes in membership of the dot pattern. For ease of reference we assign to each change identifier a label in SMALL CAPITALS.

### 3.1 Change in centroid scaled by the bounding box: CENTROID

This change value is given by the distance between the centroids of the current dot pattern and the stored dot pattern. It is normalised by dividing it by the diagonal of the bounding box of the stored dot pattern. If the dot pattern has $n$ dots, the total computation time is $O(n)$ (If the dots are held in a suitable tree data structure, the bounding boxes can be found in time $O(\log n)$, but this does not reduce the overall order-of-magnitude complexity.)

### 3.2 Change in variance from the centroid: VARIANCE

The difference between the variances of the current and stored dot patterns.[2] We use variance rather than standard deviation so as to avoid the processing time involved in computing the square root. This measure can also be computed in time $O(n)$.

### 3.3 Change in axis-aligned medians: MEDIAN

This is given by the distance between the 'medians' of the current and stored dot patterns, where the (axis-aligned) median of a dot pat-

[2] The variance is the mean squared distance of the dots from the centroid.

tern is defined as the point whose coordinates are the medians of the x-coordinates of the dots and the y-coordinates of the dots respectively. This is analogous to the centroid but computed using the median rather than the mean. However, unlike the centroid, it is not rotation-invariant.

### 3.4 Percentage change in number of dots: DOTS

This is the difference in number of dots between the current dot pattern and the stored dot pattern as a percentage of the number of dots in the stored dot pattern. This can be computed in $O(n + i)$ time, where $i$ is the number of dots from the previous pattern.

### 3.5 Change in bounding box: BOUNDINGBOX

This is given by the area of the symmetric difference between the bounding boxes of the current and stored dot patterns; it can be computed as the sum of the areas of the bounding boxes, less twice the area of their intersection. For purposes of normalisation, it is expressed as a fraction of the area of the bounding box of the stored dot pattern. If the dots are held in a two-dimensional tree data structure, this can be computed in time $O(\log n)$.

### 3.6 Proportion of points outside the boundary of the stored footprint: OUTSIDE

The fraction of dots outside the current footprint. By using the ray-casting method [15] we can find this in $O(nm)$ time, where $m$ is the number of edges of the footprint. This is only sensibly applied if the footprint algorithm does not allow outliers, i.e. dots present in the dot pattern but not in the completed footprint. It should be noted that this is the only change identifier on our list which makes use of the stored footprint.

## 4 Change Identifier Sets

The change identifiers are used to compute the term $eval(DP_i, SDP_{i-1}, SFP_{i-1})$ used in Algorithm 1. While it is certainly possible to use any one of the change identifiers on its own for this purpose, it seems likely, given the relatively undiscriminating nature of each of them considered individually, that better results will be achieved when a group of two or more identifiers is used, with their values combined in some way to give $eval(DP_i, SDP_{i-1}, SFP_{i-1})$. In our implementation to be described below, we use an xml file to collect together change identifiers to this end. There are a two of important decisions to make in combining the change identifiers: Are they run in an order? and how are the results of the identifiers tallied? We wanted the system to allow for different value choices such that we could run multiple setups and compare how effective they are, so the xml has element types for various parameters. The identifiers all have a priority associated with them, they are then run in ascending order. The set can be set to run concurrently but it was found that for small dot sets the time taken to start the threads would be slower than the time taken to run the footprint algorithm. The xml has elements for giving different thresholds to each individual identifier, a change identifier is considered to have failed if the amount of change it returns exceeds the value of the threshold parameter. There is a total threshold parameter that is attached to the set, the identifiers' values are summed and if the result is greater than the total threshold value the set is considered to have failed and the footprint is redrawn. However, the identifiers have different scales of measurement, so that, for example, to add BOUNDINGBOX directly to VARIANCE would be to combine two very different units together and therefore may give undue importance to one identifier over another. A multiplier parameter is applied to the change value of the identifier before it is added to the total value to prevent this inequality from occurring. If the bias can not be handled by the multiplier the set can also have an integer parameter that is a threshold of how many individual identifiers are allowed to fail.

## 5 Analysis

The purpose of using change identifiers is to enable the evolution of a footprint to be tracked more efficiently than by recomputing the footprint at each time step. The footprint is only recomputed when the change identifiers indicate that the dot pattern has changed sufficiently to make the mismatch with the current stored footprint unacceptably great. The number of footprint recomputations, and hence the total time taken to process a given sequence of dot patterns, will depend on the change identifiers used, and the threshold settings. We define variables as follows:

- $t_{FP}(i)$ is the time taken to compute the footprint from the dot pattern at step $i$.
- $t_{CI}(i)$ is the time taken to evaluate the change identifiers at step $i$.
- $r(i)$ is a Boolean variable, set to 1 if the footprint is in fact recomputed at step $i$, and zero otherwise.

The total computation time over a run of $n$ dot patterns is thus

$$T_{CI} = t_{FP}(0) + \sum_{i=1}^{n} (t_{CI}(i) + r(i)t_{FP}(i)).$$

The value of $T_{CI}$ is minimum when the change identifier threshold is set so high that the footprint is never recomputed after the start of the sequence (so $r(i) = 0$ for $1 \leq i \leq n$):

$$T_{\min} = t_{FP}(0) + \sum_{i=1}^{n} t_{CI}(i).$$

It is maximum when the change identifier threshold is set so low that the footprint is recomputed at every time step (so $r(i) = 1$ for all $i$):

$$T_{\max} = t_{FP}(0) + \sum_{i=1}^{n} (t_{CI}(i) + t_{FP}(i)).$$

If change identifiers are not used at all, and the footprint recomputed at every time step, then the total time taken is

$$T_{NCI} = \sum_{i=0}^{n} t_{FP}(i) = t_{FP}(0) + T_{\max} - T_{\min}.$$

If it is assumed that always $t_{CI}(i) < t_{FP}(i)$ (for if not, there would be little point in using change identifiers) then $T_{\min} < T_{NCI} < T_{\max}$, so the relative size of $T_{CI}$ and $T_{NCI}$ — which provides a measure of the time advantage, if any, gained by using change identifiers — depends on the threshold settings.

This time advantage must be set against the accuracy with which the footprint is tracked. The cost of using change identifiers comes from the fact that, most of the time, the stored footprint differs from the true footprint. To measure this cost, we need a way of quantifying the extent of this mismatch. The difference between two footprints

can be measured in various ways, e.g., using Hausdorff distance, or symmetric area difference (see [10, Ch. 7] for a discussion). Here we will use only the symmetric area difference, but the principles described below would apply equally to other measures.

The symmetric difference between two regions comprises the parts of each region that do not overlap the other; it is given by

$$R_1 \Delta R_2 = (R_1 \setminus R_2) \cup (R_2 \setminus R_1) = (R_1 \cup R_2) \setminus (R_1 \cap R_2).$$

We use the area of this as a measure of the dissimilarity between two footprints; and since we are only interested in comparisons, not absolute values, we normalise this area by expressing it as a fraction of the area of the 'true' footprint ($FP_i$)[3]. Thus the aggregate mismatch between the stored footprint and the true footprint over a dot-pattern sequence of length $n$ is given by

$$mismatch = \sum_{i=0}^{n} \frac{||FP_i \Delta SFP_i||}{||FP_i||},$$

where $||X||$ denotes the area of region $X$.

If the footprint is recomputed every time, corresponding to total computation time $T_{\max}$, we have $SFP_i = FP_i$ for every $i$, so $mismatch = 0$. At the other extreme, the maximum value of $mismatch$ is obtained when the footprint is never recomputed, corresponding to $T_{\min}$. There is thus a trade-off between accuracy and computation time, as indicated in Figure 1, where different choices of change identifier thresholds correspond to different positions on the curve. The optimal setting for the change identifier threshold depends on the relative importance attached to the conflicting goals of minimizing both computation time and accumulated footprint error; but in any case no time advantage can be obtained for mismatches below the value $m$ at which $T_{CI} = T_{NCI}$.

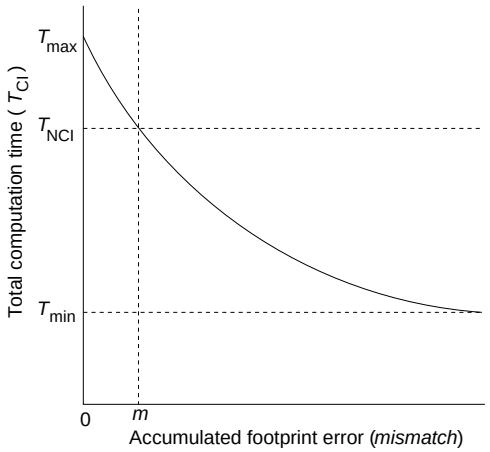



**Figure 1.** Total computation time against aggregate footprint error

# 6 Implementation

## 6.1 System

We have implemented a system to test the ideas presented in this paper. The system is split into modular parts: the engine, the change identifiers, the application and the footprint algorithms. The application initialises an instance of the engine to which it passes the footprint algorithm to use and the identifier set to process, then it starts the instance. The engine sits in a waiting state checking an internal queue to see if it has dot patterns to process. The application passes dot patterns to the engine; as in a live system it does not wait for a response but sends them continuously. The engine processes the dot patterns and notifies the application each time it generates a new stored footprint. Once the application has sent all the patterns to the engine it sends a command to stop.

If we are running this instance as a test then the engine also processes the dot patterns without using the change identifiers, recomputing the footprint every time, and records the following data:

- Time taken to run the engine over the entire set of dot patterns.
- Time taken to run the footprint algorithm.
- The state of the change identifier set at each timestep:
  - How long each change identifier took to evaluate.
  - Which change identifiers failed.
  - The value each change identifier returned.
  - What the total change was.
  - If the change identifier set enforced recomputation of the footprint, then which change identifier(s) caused the set to fail.
- Time taken to run the set.
- The current dot pattern at each time step.
- The stored footprint at each time step.
- The 'true' footprint at each time step.

By running this control we can see how much time is saved using the change identifier sets and draw similarity comparisons, giving us quantitative data to see how far the stored footprint deviates from the 'true' footprint at any time step. We use the methods described in §5 to produce two graphs: the first plots the symmetric area difference against time step, and the second plots the computation time for each time step. Results from some preliminary tests using this method are described below.

The other component of note in the system is a properties holder linked to the dot patterns. The change identifiers typically compare values computed from the current and stored dot patterns. But of course, any stored dot pattern was once current, so its value for each identifier will have been computed already. It is inefficient to compute it again so the pattern stores it in a mapping table once it is first computed.

## 6.2 Current Results

We have run tests on streams of 500 dot patterns containing up to 1000 dots each. We have implemented a collective motion pattern generator which can use different methods to produce streams of dot patterns. The method that generated the patterns for the current tests makes use of the principles of separation, cohesion and aggregation used to define behaviours in the Boids system of [14]. The footprint algorithm used is the upper and lower convex hull algorithm as given in [3]. A separate program has been written to showcase the two footprints for each timestep (one with change identifiers the other without) and time details from the test (See Figure 2).[4]

Currently the only two change identifiers for which full tests have been run are BOUNDINGBOX (§3.5) and DOTS (§3.4). Both of these

[3] We use $FP_i$ instead of $f(DP_i)$ for clarity within the formula

[4] The screenshot is from a smaller test than the one mentioned above so that the footprints are clearly visible on the small image

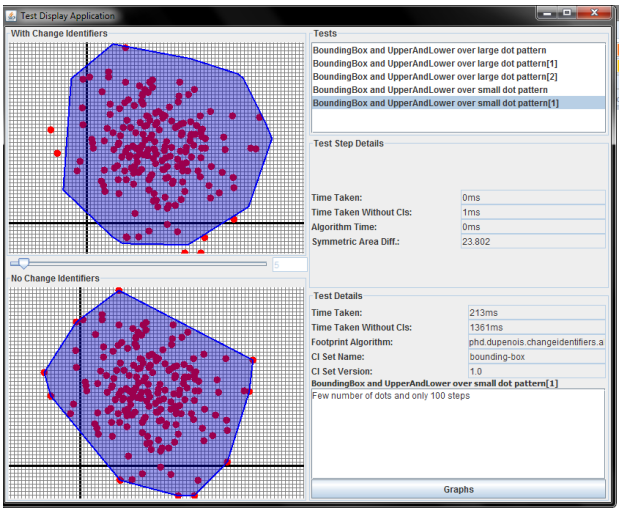


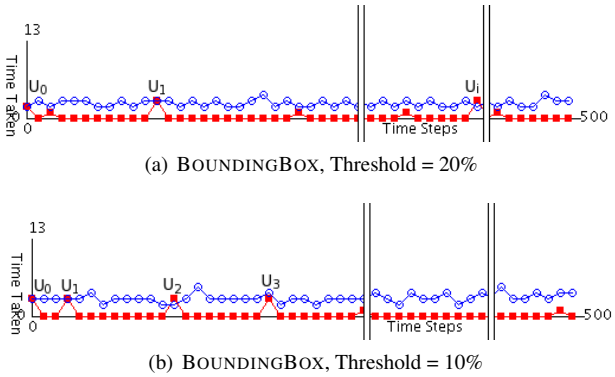

**Figure 2.** Screenshot of Result Display App.

consistently show better run times for when using change identifiers than when not, for a range of threshold settings.

The results display application produces the two graphs described in §6.1. The computation time graph (Figure 3) has two lines: The squares are on the line representing the run with change identifiers, and the circles are on the line representing the run without. The vertical double bars represent where the graph has been cut and stitched — with 500 steps the graph is too long to display in its entirety. As would be expected, the computation times when change identifiers are used is consistently less than when they are not; in fact it generally takes less than 1ms to run and therefore is less than 1ms over the footprint algorithm time when it updates. The time steps at which it updates can clearly be seen on the graph at $U_0$–$U_i$. Figure 3 shows the case where the change identifier is BOUNDINGBOX and the threshold is set at 20% (Figure 3(a)) and 10% (Figure 3(b)). The 10% threshold updates more often and we have a total time ($T_{\text{CI}}$ in Figure 1) of 90ms for the run compared to 61ms for the 20%, but both are far below the times for the comparison runs which update at each timestep, with total times ($T_{\text{NCI}}$) of 1331ms for the 20% run and 1342ms on the 10%.[5]

The symmetric area difference graph (Figure 4) also clearly shows the update times ($U_0$–$U_i$). More interesting is what it can tell us about the change of the dot pattern. The frequency with which these updates occur shows us how static the dot pattern is and, if we know the change identifier(s) used, how it changed. The area difference graph for threshold 20%(Figure 4(a)) levels out towards the end, although the cropping obscures this. This levelling out indicates that the bounding box of the dot pattern did not change by over 20% for these time steps. The area difference during this static period is around 16%; if this is within allowed footprint error then we are saving large amounts of time across the period by not updating. If, however, 16% is considered too great a footprint difference then we need to change the threshold values on the identifier set to update earlier. Figure 4(b) shows a run with the BOUNDINGBOX threshold set at 10%, and as mentioned above, this causes many more updates. Significantly, Figure 4(b) does not level out as Figure 4(a) does, showing that lowering the threshold picked up changes ignored by the larger value. The accumulated errors (as described in §5) for Figure 4(b) and Figure 4(b) are 4545.5 and 2826 respectively; these may seem large but are accumulated over 500 time steps and give us an average error of 9.091 and 5.652 per time step. Whether or not these are acceptable will depend on specific application requirements.

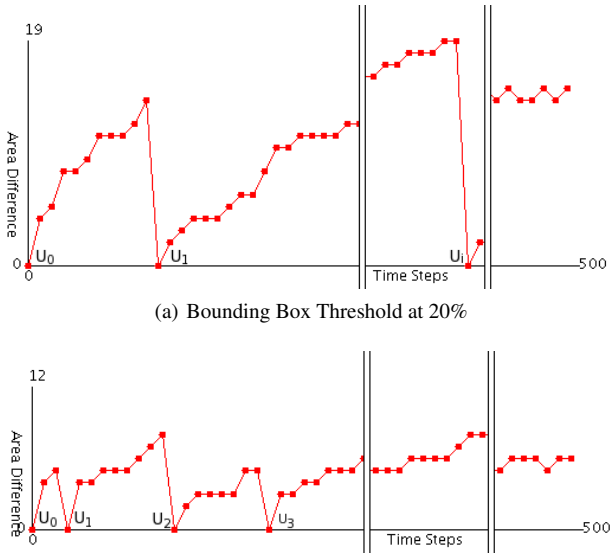
[5] Note that, in an ideal world, these values for $T_{\text{NCI}}$ would be equal, since if no change identifiers are used the difference in threshold is irrelevant; the small difference actually found merely reflects the fact that in a real-world computing environment there will always be some variation in computation times even for identical computations.




(a) BOUNDINGBOX, Threshold = 20%

(b) BOUNDINGBOX, Threshold = 10%

**Figure 3.** Graph of Time Taken against Time Steps




(a) Bounding Box Threshold at 20%

(b) Bounding Box Threshold at 10%

**Figure 4.** Graph of Footprint Area Difference against Time Steps

## 7 Conclusions and Further Work

The principles behind the change identifiers appear to be sound. The graphs show a consistent saving of 5ms per time step using only the BOUNDINGBOX change identifier. There have not yet been enough tests performed to allow a systematic comparison of the usefulness

of different change identifiers, but the change in bounding box has shown itself to be able to identify dot pattern changes and update accordingly.

The continuation of this work includes implementing the rest of the change identifiers and running basic tests on them, as with BOUNDINGBOX, to see if they affect the update times with any regularity. Once done, an application using the principles of optimisation will be created to sort through the variations of change identifier sets over a particular dot pattern stream with a particular footprint algorithm. The results of this will be plotted on a graph of accumulated footprint error against time taken, as described in §6.1. This application will need to be run over several footprint algorithms and dot pattern streams.

With regard to the different types of footprint algorithm; the $\chi$-hull algorithm from [5], the $\alpha$-shape from [7] and the swinging-arm algorithm from [9] will be implemented. The majority of non-convex footprint algorithms require some external parameter ($\alpha$ in the $\alpha$-shape, line length in the $\chi$-hull and arm length in the swinging-arm), but fortunately the selection of this parameter does not greatly concern us. Our immediate concern is with how well we can track the footprint, not how appropriate the footprint is for the dot pattern.

In [16], several types of collective movement are described, and having sets of dot pattern streams that replicate these movements would lend weight to the accuracy rating of the change identifiers. It could show that the identifier in question was accurate over all types, accurate only for some, or for none. As well as this archetypal data, we want to apply the system to real-world examples.

Another, purely qualitative, method of judging the performance of the system is to appeal to human intuition. We can record the streams as 'movies' of the footprint evolving with the dot pattern. These movies can be played to a group of experimental subjects who are asked to rate how well they felt the footprint kept up with the dot pattern. Importantly the test should be set up in such a way that the notion of a good footprint is disentangled from how well it can be tracked. Results from this experiment would indicate just how important people think accuracy is.Data from the experiment may enable us to state which change identifier sets give acceptable accuracy for high efficiency and may help us say something about what properties of the dot pattern are most important when generating a footprint.

Also of interest will be the comparison between the quantitative and the qualitative data. Comparing the accuracy assessments from the human-subject study with the results from the quantitative testing may tell us which change identifiers are most important to human intuition.

Other accuracy measures, e.g., Hausdorff distance, will also be implemented, and it will be interesting to see how they relate to each other. A side interest will be to see how they relate to the accuracy ratings from the human study, as it may be that one of the measures is, implicitly, more used by the human mind than others.

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